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AXISYMMETRIC MIXED BOUNDARY VALUE PROBLEMS FOR AN ELASTIC HALFSPACE WITH A PERIODIC NONHOMOGENEITY

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Abstract—This paper examines certain axisymmetric contact, crack and inclusion problems related to a nonhomogeneous elastic medium where the two elastic parameters are periodic functions of the axial variable z. A general formulation of the equations governing the axisymmetric deformations of the medium is presented. It is shown that the mixed boundary value problems can be reduced to a set of two ordinary differential equations and a Fredholm integral equation of the second kind. The ordinary differential equations are solved in a numerical fashion, and these solutions are used to evaluate the kernel function of the Fredholm integral equation. The resulting integral equations governing the contact problem are also solved via a numerical technique to obtain the load–displacement relationship for a rigid circular indentor in smooth contact with a periodically non-homogeneous elastic halfspace region. The procedures are extended to examine the problems associated with a penny-shaped crack and a rigid disc inclusion embedded in such a non-homogeneous elastic medium. © 1998 Elsevier Science Ltd.

1. INTRODUCTION

The classical theory of elasticity has been successfully applied to examine a variety of contact problems of technological interest (Galin, 1961; Ufliand, 1965; Selvadurai, 1979; Gladwell 1980; Johnson, 1985). In classical studies, the emphasis is on the modeling of the homogeneous medium either as an isotropic elastic or transversely isotropic elastic solid. Extensive studies related to halfspace regions indented by axisymmetric and nonsymmetric indentors are documented in the references cited previously. These studies have also been extended to include frictional constraints at the contact region (Spence, 1968; de Pater and Kalker, 1975; Turner, 1979).

Contact problems involving nonhomogeneous elastic media are generally regarded as non-classical problems in the theory of elasticity. In these problems, the isotropic or anisotropic elastic constants are assumed to be functions of the spatial variables. The applications of the theory of elasticity for a nonhomogeneous elastic material for the solution of contact and other traction boundary value problems are given by Korenev (1957), Mossakovskii (1958), Rakov and Rvachev (1961), Belik and Protsenko (1967), Gibson (1967), Stachowicz (1968), Gibson *et al.* (1971), Awojobi and Gibson (1973), Kassir and Chuaprasert (1974), Gibson and Sills (1975) and Selvadurai *et al.* (1986). Recently, Selvadurai (1996) has applied the theory of elasticity for a nonhomogeneous elastic medium to examine contact problems related to nonhomogeneous elastic media where only the linear elastic shear modulus varies exponentially with the axial coordinate and is finitely bounded within a halfspace region. Selvadurai (1996) also gives an extensive review of the application of the theory of elasticity for a nonhomogeneous elastic medium to a variety of contact problems and other traction boundary value problems where the elastic constants vary with the axial coordinate.

This paper examines the problem of the axisymmetric indentation of the surface of a nonhomogeneous elastic solid in which the elastic properties are assumed to be *harmonic functions* of the axial coordinate. This type of elastic nonhomogeneity is a useful approximation for modeling certain problems of technological interest. For example, in connection with laminated materials consisting of alternate layers of isotropic materials, the diffusion of adherents and chemicals can initiate changes in the elastic properties (both increases and decreases in the stiffness characteristics) in the separate layers. Alternatively, composites could be fabricated with surface treated layers which will have the appropriate variations. Also, in the context of geological media, the periodic elastic nonhomogeneity serves as a useful approximation for the study of sedimentary geological materials with a depositional history (Tschebotarioff, 1973).

This paper presents the general formulation of the problem governing the axisymmetric deformations of the medium where the two elastic parameters are functions of the axial variable z. It is shown that the mixed boundary value problems associated with contact, crack and inclusion problems can be reduced to a set of two ordinary differential equations and a Fredholm integral equation of the second kind. The ordinary differential equations are solved by the differential equation solver COLSYS, and these solutions are used to evaluate the kernel function of the Fredholm integral equation. The Fredholm integral equation governing the mixed boundary value problem is then solved numerically to evaluate the load–displacement relationship for the rigid circular indentor. These numerical results illustrate the manner in which the periodic variation, and in particular a harmonic variation in the linear elastic shear modulus, influences the axial stiffness of the rigid indentor. The procedures are extended to examine the mixed boundary value problems associated with a penny-shaped crack and a rigid disc inclusion embedded in such a nonhomogeneous elastic medium of infinite extent.

2. FUNDAMENTAL EQUATIONS

We restrict our attention to axisymmetric deformation of the nonhomogeneous elastic medium which is characterized by the displacement vector

$$u_i = (u_r, 0, u_z) \tag{1}$$

which is referred to a cylindrical coordinate system. The non-zero components of the strain tensor ε_{ij} are given by

$$\varepsilon_{ij} = \begin{bmatrix} \dfrac{\partial u_r}{\partial r} & 0 & \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right) \\ 0 & \dfrac{u_r}{r} & 0 \\ \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right) & 0 & \dfrac{\partial u_z}{\partial z} \end{bmatrix}. \tag{2}$$

The state of stress in the elastic medium is given by the stress tensor σ_{ij} referred to the cylindrical coordinate system;

$$\sigma_{ij} = \begin{bmatrix} \sigma_{rr} & 0 & \sigma_{rz} \\ 0 & \sigma_{\theta\theta} & 0 \\ \sigma_{rz} & 0 & \sigma_{zz} \end{bmatrix}. \tag{3}$$

The linear elastic stress–strain relationship for the nonhomogeneous elastic medium takes the form

$$\sigma_{ij} = 2G(z)\varepsilon_{ij} + \frac{2\nu(z)G(z)}{1-2\nu(z)}\varepsilon_{kk}\delta_{ij} \tag{4}$$

where $\varepsilon_{kk} = \varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{zz} = e$; δ_{ij} is Kronecker's delta function and $G(z)$ and $\nu(z)$ are, respectively, the linear elastic shear modulus and Poisson's ratio which depend only on the axial coordinate z. Formulation presented in this section applies to general variations of $G(z)$ and $\nu(z)$. Using eqns (2) and (4), the non-trivial equations of equilibrium can be expressed in the form

$$\nabla^2 u_r + \frac{1}{1-2\nu}\frac{\partial e}{\partial r} - \frac{u_r}{r^2} + \frac{1}{G}\frac{\mathrm{d}G}{\mathrm{d}z}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) = 0 \tag{5}$$

$$\nabla^2 u_z + \frac{1}{1-2\nu}\frac{\partial e}{\partial z} + \frac{2}{G}\frac{\mathrm{d}G}{\mathrm{d}z}\left(\frac{\partial u_z}{\partial z} + \frac{\nu e}{1-2\nu}\right) + e\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{2\nu}{1-2\nu}\right] = 0 \tag{6}$$

where ∇^2 is the axisymmetric form of Laplace's operator referred to the cylindrical coordinate system, i.e.

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{7}$$

For the solution of the displacement equations of equilibrium (5) and (6), we introduce Hankel transform representations of the displacement components (Sneddon, 1951)

$$u_r(r,z) = \int_0^\infty U(s,z)A(s)J_1(rs)\,\mathrm{d}s \tag{8}$$

$$u_z(r,z) = \int_0^\infty W(s,z)A(s)J_0(rs)\,\mathrm{d}s \tag{9}$$

where $A(s)$ is an arbitrary function and $J_n(rs)$ is the nth order Bessel function of the first kind. Using the above integral representations for the displacement components, the equilibrium eqns (5) and (6) can be reduced to the following forms

$$\frac{\mathrm{d}^2 U}{\mathrm{d}z^2} + q(z)\frac{\mathrm{d}U}{\mathrm{d}z} - \frac{2(1-\nu)}{1-2\nu}s^2 U - \frac{s}{1-2\nu}\frac{\mathrm{d}W}{\mathrm{d}z} - q(z)sW = 0 \tag{10}$$

$$\frac{\mathrm{d}^2 W}{\mathrm{d}z^2} + [p(z)+q(z)]\frac{\mathrm{d}W}{\mathrm{d}z} - \frac{1-2\nu}{2(1-\nu)}s^2 W + \frac{s}{2(1-\nu)}\frac{\mathrm{d}U}{\mathrm{d}z} + [p(z) + \frac{\nu}{1-\nu}q(z)]sU = 0 \tag{11}$$

where

$$p(z) = \left[\frac{1-2\nu(z)}{1-\nu(z)}\right]\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\nu(z)}{1-2\nu(z)}\right], \quad q(z) = \frac{1}{G(z)}\frac{\mathrm{d}G(z)}{\mathrm{d}z}. \tag{12}$$

In order to formulate the mixed boundary value problem related to the contact, crack and inclusion problems, we require expressions for the stress components σ_{zz} and σ_{rz}. Substituting the expressions (8) and (9) into eqns (2) and (4), we obtain

$$\sigma_{zz}(r, z) = \frac{2G(z)[1-\nu(z)]}{1-2\nu(z)} \int_0^\infty \left[\frac{\mathrm{d}W}{\mathrm{d}z} + \frac{\nu(z)sU}{1-\nu(z)}\right] A(s) J_0(rs)\,\mathrm{d}s \tag{13}$$

$$\sigma_{rz} = G(z) \int_0^\infty \left(\frac{\mathrm{d}U}{\mathrm{d}z} - sW\right) A(s) J_1(rs)\,\mathrm{d}s. \tag{14}$$

3. THE INDENTATION PROBLEM

In this section we consider the problem of the smooth indentation of the surface of a nonhomogeneous elastic halfspace region by a circular indentor of radius a (see Fig. 1). The problem is axisymmetric and the mixed boundary conditions associated with the indentation are as follows:

$$u_z(r, 0) = \Delta, \quad r \leqslant a \tag{15}$$

$$\sigma_{zz}(r, 0) = 0, \quad r > a \tag{16}$$

$$\sigma_{rz}(r, 0) = 0, \quad r \geqslant 0. \tag{17}$$

In addition to the above boundary conditions at $z = 0$, the displacement and stress fields should satisfy the regularity conditions, $u_i \to 0$ and $\sigma_{ij} \to 0$ as $z \to \infty$. Considering these conditions and the expressions (8) and (9) for the displacement components u_r and u_z, we have

$$U(s, \infty) = W(s, \infty) = 0. \tag{18}$$

The shear stress boundary condition (17) gives the result

$$\left[\frac{\mathrm{d}U}{\mathrm{d}z} - sW\right]_{z=0} = 0. \tag{19}$$

To ensure a well-posed boundary value problem for the ordinary differential eqns (10) and (11), a further boundary condition is required at $z = 0$. Considering the linearity of these two equations, we can impose, without any loss of generality, the following regularity condition on W

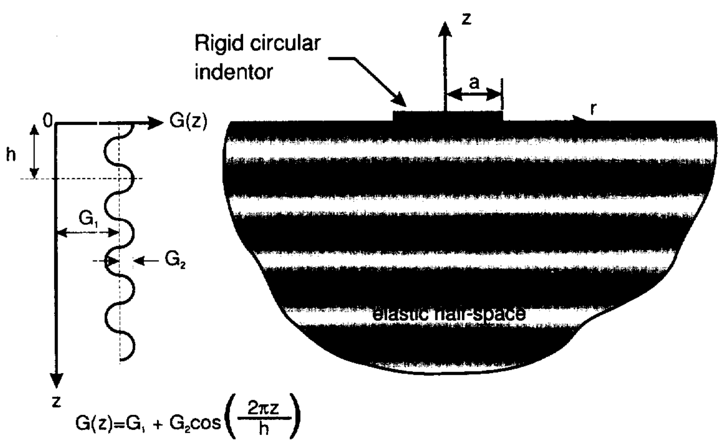

Fig. 1. Rigid circular indentor in smooth contact with a nonhomogeneous elastic halfspace region.

$$W(s,0) = 1. \tag{20}$$

It can be shown that the mixed boundary conditions (15) and (16) and the relevant integral expressions for u_z and σ_{zz} lead to the following system of dual integral equations

$$\int_0^\infty W(s,0)A(s)J_0(rs)\,\mathrm{d}s = \Delta\,; \quad r \leqslant a \tag{21}$$

$$\int_0^\infty \left[\frac{\mathrm{d}W}{\mathrm{d}z} + \frac{\nu s U}{1-\nu}\right]_{z=0} A(s)J_0(rs)\,\mathrm{d}s = 0\,; \quad r > a. \tag{22}$$

Once solutions are obtained for U and W, the system of integral eqns (21) and (22) are dual integral equations for the function $A(s)$. By introducing the substitutions

$$sR(s) = \left[\frac{\mathrm{d}W}{\mathrm{d}z} + \frac{\nu s U}{1-\nu}\right]_{z=0} \quad \text{and} \quad B(s) = A(s)R(s) \tag{23}$$

we can rewrite the dual system (21) and (22) as

$$\int_0^\infty \left[\frac{W}{R(s)}\right]_{z=0} B(s)J_0(rs)\,\mathrm{d}s = \Delta\,; \quad r \leqslant a \tag{24}$$

$$\int_0^\infty sB(s)J_0(rs)\,\mathrm{d}s = 0\,; \quad r > a. \tag{25}$$

Considering the finite Fourier transform representation for $B(s)$ in the form

$$B(s) = \frac{2\Delta}{\pi}\int_0^a \phi(t)\cos(st)\,\mathrm{d}t \tag{26}$$

it is evident that the traction boundary condition (25) is automatically satisfied and the displacement boundary condition (24) gives rise to a single integral equation of the form

$$\int_0^a K(x,t)\phi(t)\,\mathrm{d}t = \frac{\pi}{2} \tag{27}$$

where the kernel function $K(x, t)$ is given by

$$K(x,t) = \int_0^\infty \left[\frac{W}{R(s)}\right]_{z=0} \cos(st)\cos(sx)\,\mathrm{d}s. \tag{28}$$

Upon solving the ordinary differential equations (10) and (11), the kernel function (28) can be evaluated and the integral equation (27) can then be solved by employing a standard numerical technique [see e.g. Delves and Mohamed (1985)].

For the contact problem, the total force and indentation relationship is of particular interest to technological applications. With given indentation Δ, the total force P_c can be evaluated by considering the equilibrium of the indentor, i.e.

$$P_c = 2\pi\int_0^a r\sigma_{zz}(r,0)\,dr = \frac{8G(0)\Delta[1-\nu(0)]}{1-2\nu(0)}\int_0^a \phi(t)\,dt. \tag{29}$$

It is also of interest to examine the reduction to the limiting case of a homogeneous isotropic elastic solid which is defined by the limits $G(z)$ = const = G and $\nu(z)$ = const = ν. In this case $p(z) = q(z) = 0$, and the differential equations (10) and (11) give the solutions

$$U(s,z) = \frac{1}{2(1-\nu)}[(2\nu-1)+sz]e^{-sz} \tag{30}$$

$$W(s,z) = \left[1+\frac{sz}{2(1-\nu)}\right]e^{-sz} \tag{31}$$

which lead to the well-known Boussinesq's result for the load–displacement relationship for a rigid circular indentor in smooth contact with a homogeneous elastic halfspace region with elastic constants $G = G_1$ and ν [see, e.g. Gladwell (1980)]

$$P_{c0} = \frac{4G_1\Delta a}{1-\nu} \tag{32}$$

where we have assumed that Δ occurs in the direction of the applied force P_{c0}. Note that in the homogeneous case $[W/R(s)]_{z=0}$ is a function of ν only and the kernel function $K(x,t)$ given by (28) is a delta function.

4. NUMERICAL SCHEME AND RESULTS

It is unlikely that we can find exact closed form solutions or analytical solutions for the set of two ordinary differential equations for arbitrary choices of nonhomogeneity. Consequently, it is necessary to solve them numerically. In this section we present some of the key features of the numerical scheme, which are followed by numerical results for the indentation problem related to a periodically nonhomogeneous halfspace.

By introducing the substitutions

$$S = sa, \quad Z = \frac{z}{a} \tag{33}$$

and a non-dimensional variable

$$X = sz = SZ \tag{34}$$

the equations governing the displacement variables U and W can be rewritten as

$$\frac{d^2U}{dX^2}+Q(X)\frac{dU}{dX}-\frac{2(1-\nu)}{1-2\nu}U-\frac{1}{1-2\nu}\frac{dW}{dX}-Q(X)W = 0 \tag{35}$$

$$\frac{d^2W}{dX^2}+[P(X)+Q(X)]\frac{dW}{dX}-\frac{1-2\nu}{2(1-\nu)}W+\frac{1}{2(1-\nu)}\frac{dU}{dX}+\left[P(X)+\frac{\nu Q(X)}{1-\nu}\right]U = 0 \tag{36}$$

where

$$P(X) = \left[\frac{1-2\nu}{1-\nu}\right]\frac{\mathrm{d}}{\mathrm{d}X}\left[\frac{\nu}{1-2\nu}\right], \quad Q(X) = \frac{1}{G}\frac{\mathrm{d}G}{\mathrm{d}X}. \tag{37}$$

Using these substitutions and $T = t/a$, we can rewrite the Fredholm integral equation in the following dimensionless form

$$\int_0^1 K(X,T)\phi(T)\,\mathrm{d}T = \frac{\pi}{2} \tag{38}$$

where

$$K(X,T) = \int_0^\infty \left[\frac{W}{\dfrac{\mathrm{d}W}{\mathrm{d}X}+\dfrac{\nu U}{1-\nu}}\right]_{X=0} \cos(ST)\cos(SX)\,\mathrm{d}S. \tag{39}$$

In the ensuing, we shall restrict attention to the consideration of a nonhomogeneous elastic halfspace region where only the linear elastic shear modulus varies according to a harmonic relationship and Poisson's ratio is a constant; i.e.

$$G(z) = G_1 + G_2\cos\left(\frac{2\pi z}{h}\right), \quad \nu(z) = \nu = \text{const} \tag{40}$$

where h is the periodicity in the variation of the shear modulus in z-direction. For this particular nonhomogeneity, we have

$$P(X) = 0, \quad Q(X) = -\frac{2\pi\chi a\sin\left(\dfrac{2\pi aX}{hS}\right)}{hS\left[1+\chi\cos\left(\dfrac{2\pi aX}{hS}\right)\right]} \tag{41}$$

where $\chi = G_2/G_1$. It should be noted that, the specific choices for G_1, G_2 and ν can only be assigned values which will ensure *pointwise positive definiteness* of the strain energy function in the domain of interest. In general, it is noted that the positive definiteness constraints take the form

$$G(z) > 0, \quad -1 < \nu(z) < 1/2 \tag{42}$$

for the halfspace region with $r \in (0,\infty)$, $z \in (0,\infty)$.

The set of differential equations (35) and (36) can be solved numerically by using the boundary value problem solver COLSYS described by Ascher *et al.* (1981). Note that the domain for these differential equations is $[0,\infty)$. In order to take full advantage of the collocation procedure used in COLSYS, we introduce the following change of variable

$$X = \tan\left(\frac{\pi}{2}Y\right) \tag{43}$$

which transforms the infinite domain $(0,\infty)$ to a finite domain $[0,1)$. With this transformation, eqns (35) and (36) are reduced to the following

$$\frac{d^2U}{dY^2}+[C_2+C_1Q(Y)]\frac{dU}{dY}-\frac{2(1-\nu)}{1-2\nu}C_1^2U-\frac{C_1}{(1-2\nu)}\frac{dW}{dY}-C_1^2Q(Y)W=0 \quad (44)$$

$$\frac{d^2W}{dY^2}+\{C_2+C_1[P(Y)+Q(Y)]\}\frac{dW}{dY}$$
$$-\frac{1-2\nu}{2(1-\nu)}C_1^2W+\frac{1}{2(1-\nu)}C_1\frac{dU}{dY}+C_1^2\left[P(Y)+\frac{\nu Q(Y)}{(1-\nu)}\right]U=0 \quad (45)$$

where

$$C_1=\frac{\pi}{2}\left[1+\tan^2\left(\frac{\pi}{2}Y\right)\right] \quad (46)$$

$$C_2=-\pi\tan\left(\frac{\pi}{2}Y\right). \quad (47)$$

Also we note that, from the definition of the kernel function (39), the solutions of U, W and their derivatives are required only at one end, $X = 0$, of the interval. Once U, W and dW/dX are known at $X = 0$, the kernel function (39) can be evaluated and the Fredholm integral equation (38) can then be solved.

The accuracy of the numerical evaluations of the integral equation depends not only on accuracy with which the solutions of the differential equations can be found, but also on the accuracy with which the kernel function $K(X, T)$ can be evaluated. The evaluation of the kernel function which contains an oscillatory integrand is approached by using a technique which takes full advantage of the periodicity of the oscillatory factor in the integrand. By using this method, the integral of any function $f(x)$, which is integrable on the infinite domain $[0, \infty)$, can be reduced to an integral of a new function $g(y)$ over a finite domain $[0, 1]$:

$$\int_0^\infty f(s)\cos(as)\,ds=\sum_{k=0}^{\infty}\int_{2k\pi/a}^{2(k+1)\pi/a}f(s)\cos(as)\,ds=\frac{2\pi}{a}\int_0^1 g(y)\cos(2\pi y)\,dy \quad (48)$$

where

$$g(y)=\sum_{k=0}^{\infty}f\left[\frac{2\pi}{a}(y+k)\right] \quad (49)$$

is a periodic function with period of unity. Considering that

$$\lim_{S\to\infty}\left[\frac{W}{\dfrac{dW}{dX}+\dfrac{\nu U}{1-\nu}}\right]_{X=0}=\frac{2(1-\nu)^2}{2\nu-1} \quad (50)$$

we can rewrite the function as

$$\left[\frac{W}{\dfrac{dW}{dX}+\dfrac{\nu U}{1-\nu}}\right]_{X=0}=\frac{2(1-\nu)^2}{2\nu-1}+F(S). \quad (51)$$

Therefore, the kernel function $K(X, T)$ can be expressed in terms of $F(S)$ and the Fredholm integral equation of the first kind can be represented as a Fredholm equation of the second kind

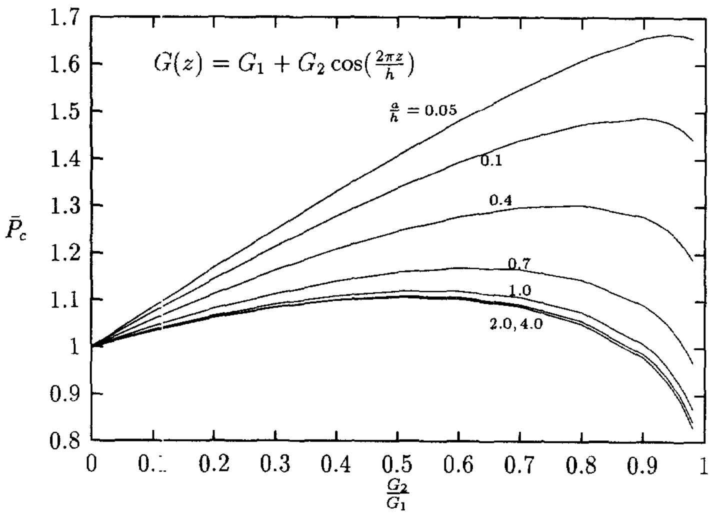

Fig. 2. Contact problem: variation of the nondimensional force $\bar{P}_c$ with respect to the ratio G_2/G_1 for $\nu = 0.3$ and various values of a/h.

$$\frac{2(1-\nu)^2}{2\nu-1}\phi(X)+\int_0^1 K^*(X,T)\phi(T)\,dT = 1 \tag{52}$$

where

$$K^*(X,T) = \frac{1}{\pi}\int_0^\infty F(S)\cos[(X+T)S]\,dS+\frac{1}{\pi}\int_0^\infty F(S)\cos[(X-T)S]\,dS. \tag{53}$$

The integral equation (52) can now be solved by using a quadrature technique similar to that outlined by Delves and Mohamed (1985).

The numerical procedures, outlined previously, are used to determine the function ϕ, which in turn could be used to evaluate the load–displacement behavior for the rigid circular indentor. Only the final results of the numerical analysis are presented in Fig. 2, in which $\bar{P}_c = P_c/P_{c0}$. From Fig. 2 we can see that the total force $\bar{P}_c$ required for a given indentation reduces dramatically if local defects (where $G(z) \to 0$ when $G_2/G_1 \to 1$ for some z) exist. Figure 2 also shows us that $\bar{P}_c$ generally decreases when a/h increases.

5. APPLICATIONS TO CRACK AND INCLUSION PROBLEMS

The mathematical and numerical procedures outlined in the previous section can also be adopted in a very straightforward way to examine other axisymmetric mixed boundary value problems associated with nonhomogeneous elastic media. The governing integral equations and numerical results for these problems are summarized here for completeness.

5.1. *A disc inclusion problem*

We examine the problem of a rigid circular disc inclusion of radius a which is embedded in bonded contact with a nonhomogeneous elastic medium of infinite extent with a constant Poisson's ratio and a harmonic variation in the linear elastic shear modulus which is symmetric about $z = 0$ (see Fig. 3). The inclusion is subjected to an axisymmetric force P_I which induces a rigid displacement Δ in the z-direction. Owing to the asymmetry in the deformations about the plane $z = 0$, the integral equation problem can be formulated as a mixed boundary value problem applicable to a halfspace region ($z \geqslant 0$) with the following mixed boundary conditions

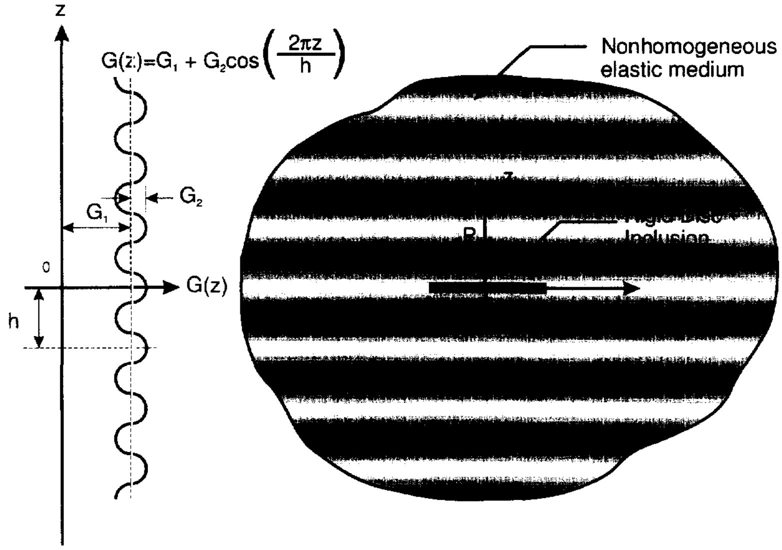

Fig. 3. Rigid circular disc inclusion embedded at a nonhomogeneous elastic interface.

$$u_z(r,0) = w(r) = \Delta; \quad r \leqslant a \tag{54}$$

$$\sigma_{zz}(r,0) = 0; \quad r \geqslant a \tag{55}$$

$$u_r(r,0) = 0; \quad r \geqslant 0. \tag{56}$$

It must be emphasized that these boundary conditions are applicable to distributions of $G(z)$ which are symmetric about $z = 0$. This requirement is automatically satisfied by the variation of $G(z)$ given by (40).

As with the indentation problem, the disc inclusion problem can be formulated as a set of ordinary differential equations and a Fredholm integral equation of the second kind. The differential equations are the same as eqns (35) and (36) for the indentation problem, but with different boundary conditions. The Fredholm integral equation takes the form

$$\frac{3-4\nu}{2(2\nu-1)}\psi(x)+\int_0^a K_1(x,t)\psi(t)\,\mathrm{d}t = 1 \tag{57}$$

where

$$K_1(x,t) = \frac{2}{\pi}\int_0^\infty \left\{\left[\frac{W}{R}\right]_{z=0} - \frac{3-4\nu}{2(2\nu-1)}\right\}\cos(sx)\cos(st)\,\mathrm{d}s. \tag{58}$$

The force–displacement relationship for the rigid disc inclusion can be obtained by considering the equilibrium of the inclusion, i.e.

$$P_1 = 2\pi\int_0^a [\sigma_{zz}(r,0^+)-\sigma_{zz}(r,0^-)]r\,\mathrm{d}r = \frac{16G(0)\Delta(1-\nu)}{1-2\nu}\int_0^a \psi(t)\,\mathrm{d}t \tag{59}$$

where 0^+ and 0^- refer to surfaces of the disc inclusion in contact with the halfspace region $z > 0$ and $z < 0$, respectively.

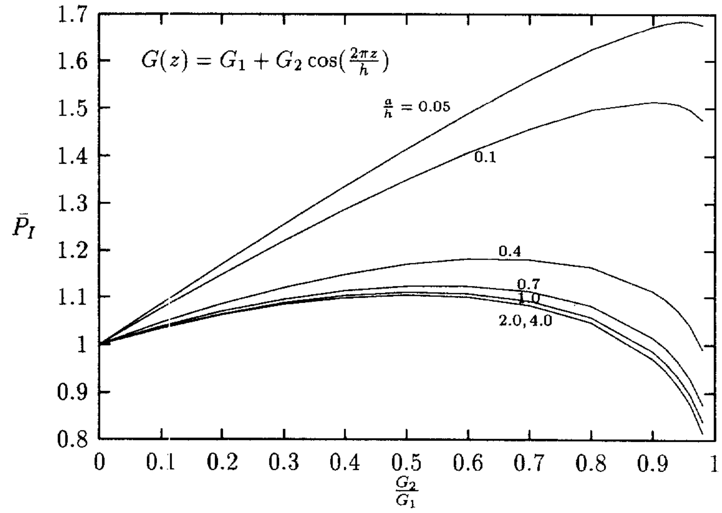

Fig. 4. Inclusion problem: variation of the nondimensional force $\bar{P}_{\rm I}$ with respect to the ratio G_2/G_1 for $\nu = 0.3$ and various values of a/h.

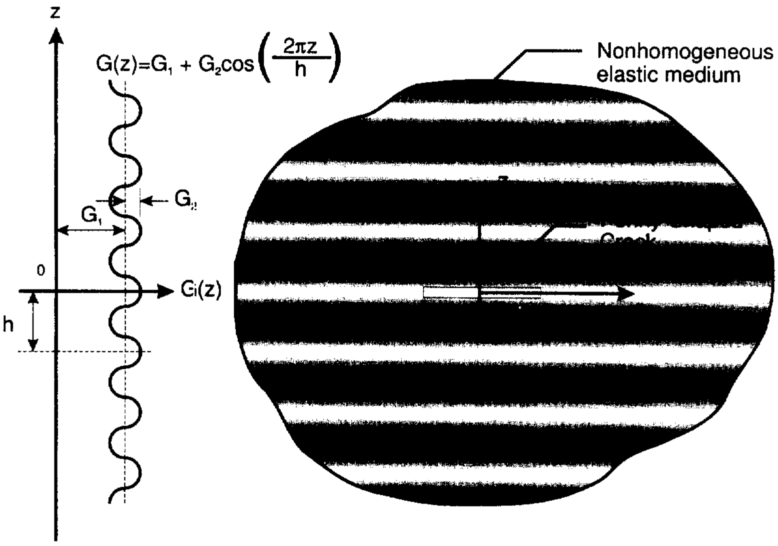

Fig. 5. A penny-shaped crack in an elastic medium with a periodic nonhomogeneity.

Numerical results for the stiffness of the disc inclusion, applicable to the periodic variation in the linear elastic shear modulus (40) are presented in Fig. 4. In these representations the nondimensional force $\bar{P}_{\rm I}$ is given by

$$\bar{P}_{\rm I} = \frac{P_{\rm I}}{P_{\rm I0}} \tag{60}$$

and $P_{\rm I0} = 32G_1 a\Delta(1-\nu)/(3-4\nu)$ is the total force required to induce a displacement Δ in a disc inclusion which is embedded in a homogeneous elastic infinite space region with a constant shear modulus G_1.

5.2. *A penny-shaped crack problem*

We examine the problem of a penny-shaped crack of radius a, which is located in an isotropic elastic medium with a periodic variation in the linear elastic shear modulus and constant Poisson's ratio (see Fig. 5). The nonhomogeneity is characterized by (40) which restricts symmetric distributions of $G(z)$ about $z = 0$. It is assumed that the crack is located at the plane $z = 0$. The infinite medium containing the penny-shaped crack is subjected to a uniform far field axial tensile stress σ_0. Owing to the symmetry of the crack problem about $z = 0$, it is possible to formulate the crack problem as a mixed boundary value problem applicable to the halfspace region $z \geqslant 0$. The mixed boundary conditions governing this problem are as follows

$$\sigma_{rz}(r,0) = 0\,; \quad r > 0 \tag{61}$$

$$\sigma_{zz}(r,0) = -\sigma_0\,; \quad r < a \tag{62}$$

$$u_z(r,0) = 0\,; \quad r \geqslant a. \tag{63}$$

The crack problem can also be reduced to a set of ordinary differential equations, which are identical to those for the indentation problem, and a Fredholm integral equation, which takes the form

$$\frac{(2\nu-1)}{2(1-\nu)^2}\chi(x) + \int_0^a K_2(x,t)\chi(t)\,\mathrm{d}t = -\frac{2x}{\pi} \tag{64}$$

where

$$K_2(x,t) = \frac{2}{\pi}\int_0^\infty \left\{\left[\frac{R}{W}\right]_{X=0} - \frac{(2\nu-1)}{2(1-\nu)^2}\right\}\sin(sx)\sin(st)\,\mathrm{d}s. \tag{65}$$

A result of some importance to fracture mechanics concerns the evaluation of the crack opening mode stress intensity factor at the crack tip. It can be shown that the non-homogeneous material property described in eqn (40) does not affect the $(r-a)^{-1/2}$ stress singularity as obtained for the homogeneous material, i.e.

$$\sigma_{zz}(r,0) = \frac{K_\mathrm{I}}{\sqrt{2(r-a)}} + O([r-a]^0). \tag{66}$$

The stress intensity factor K_I, however, is altered depending upon G_1, G_2 and ν through $\chi(a)$, i.e.

$$K_\mathrm{I} = \frac{(2\nu-1)\sigma_0\chi(a)}{2(1-\nu)^2\sqrt{a}}. \tag{67}$$

In Fig. 6 we present results for the nondimensional mode I stress intensity factor at the tip

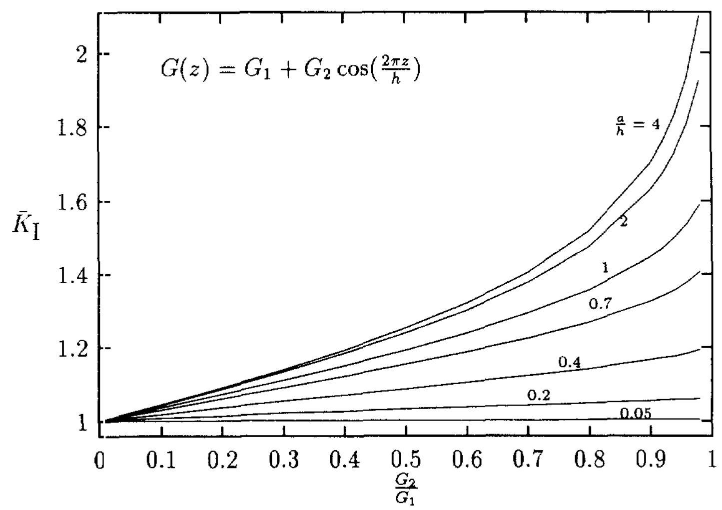

Fig. 6. Crack problem: variation of the nondimensional stress intensity factor $\bar{K}_\mathrm{I}$ with respect to G_2/G_1 for $\nu = 0.3$ and various values of a/h.

of the penny-shaped crack located in a nonhomogeneous medium with shear modulus given by (40). The non-dimensional stress intensity factor $\bar{K}_I$ is defined as

$$\bar{K}_I = \frac{K_I}{K_{I0}} \tag{68}$$

where $K_{I0} = 2\sigma_0\sqrt{a/\pi}$ corresponds to the mode I stress intensity factor for a penny-shaped crack located in a homogeneous medium, which is subjected to a far field axial stress. We note that K_{I0} is independent of the elastic constants of the medium.

6. CONCLUSIONS

The periodic elastic nonhomogeneity considered in this paper is intended to model the mechanical behavior of laminated materials where the boundaries between laminations can experience alterations in the elastic properties. The paper develops the mathematical formulation of a class of axisymmetric mixed boundary value problems related to indentation, inclusion and crack problems. In all these cases the problems are reduced to the solution of a Fredholm integral equation of the second kind. The kernel functions associated with these integral equations are derived from the solution of a pair of coupled ordinary differential equations. The accuracy in the numerical solution of these differential equations is a prerequisite for the accurate numerical solution of the Fredholm integral equations. The numerical procedures outlined in the paper can be utilized to evaluate results of specific interest to technological applications. These include the evaluation of the stiffness of an indentor resting in smooth contact with an elastic halfspace, the compliance of a disc inclusion embedded in bonded contact with an elastic infinite space and the stress intensity factor at the tip of a penny-shaped crack. It is shown that the results for all these situations can be evaluated to illustrate the influence of the periodic nonhomogeneity. It is shown that the influence of the periodic nonhomogeneity on the compliance of either the indentor or the inclusion becomes significant when the periodicity of the homogeneity (h) is large in comparison to the radius (a) of the indentor or the inclusion. For the case of the penny-shaped crack, the mode I stress intensity factor at the crack tip increases with G_2/G_1 and the nondimensional ratio a/h. The analysis and the numerical procedures presented in the paper are sufficiently general that the influence of the variability in Poisson's ratio can be readily incorporated in the numerical treatments. Similar conclusions apply to the examination of contact problems where the axisymmetric indentor has an arbitrary profile.

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